A few other bits of matrix notation and facts

$\mathbf{AB} \neq \mathbf{BA}$. Other exceptions are associated with zero matrices. A **zero matrix** is one whose elements are *all* zero, such as

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We ordinarily denote a zero matrix (whatever its size) by $\mathbf{0}$. It should be clear that for any matrix $\mathbf{A}$,

$$\mathbf{0} + \mathbf{A} = \mathbf{A} = \mathbf{A} + \mathbf{0}, \quad \mathbf{A0} = \mathbf{0}, \quad \text{and} \quad \mathbf{0A} = \mathbf{0},$$

where in each case $\mathbf{0}$ is a zero matrix of appropriate size. Thus zero matrices appear to play a role in the arithmetic of matrices similar to the role of the real number 0 in ordinary arithmetic.

Recall that an *identity matrix* is a square matrix $\mathbf{I}$ that has ones on its principal diagonal and zeros elsewhere. Identity matrices play a role in matrix arithmetic which is strongly analogous to that of the real number 1, for which $a \cdot 1 = 1 \cdot a = a$ for all values of the real number a. For instance, you can check that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Similarly, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$. For instance, the element in the second row and third column of $\mathbf{AI}$ is

$$(a_{21})(0) + (a_{22})(0) + (a_{23})(1) = a_{23}.$$

Recall that the $n \times n$ **identity matrix** is the diagonal matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{1}$$

having ones on its main diagonal and zeros elsewhere. It is not difficult to deduce directly from the definition of the matrix product that $\mathbf{I}$ acts like an identity for matrix multiplication:

$$\mathbf{AI} = \mathbf{A} \quad \text{and} \quad \mathbf{IB} = \mathbf{B} \tag{2}$$

if the sizes of $\mathbf{A}$ and $\mathbf{B}$ are such that the products $\mathbf{AI}$ and $\mathbf{IB}$ are defined. It is, nevertheless, instructive to derive the identities in (2) formally from the two basic facts about matrix multiplication that we state below. First, recall that the notation

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_n \end{bmatrix} \tag{3}$$

expresses the $m \times n$ matrix $\mathbf{A}$ in terms of its column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$.

Fact 1 **Ax** *in terms of columns of* **A**

If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an n-vector, then

$$\mathbf{Ax} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n. \tag{4}$$

The reason is that when each row vector of $\mathbf{A}$ is multiplied by the column vector $\mathbf{x}$, its jth element is multiplied by x_j.

Fact 2 **AB** *in terms of columns of* **B**

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$ is an $n \times p$ matrix, then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix}. \tag{5}$$

That is, *the jth column of* **AB** *is the product of* **A** *and the jth column of* **B**. The reason is that the elements of the jth column of $\mathbf{AB}$ are obtained by multiplying the individual rows of $\mathbf{A}$ by the jth column of $\mathbf{B}$.

Example 1 The third column of the product **AB** of the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 7 & 5 & -4 \\ -2 & 6 & 3 & 6 \\ 5 & 1 & -2 & -1 \end{bmatrix}$$

is

$$\mathbf{Ab}_3 = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}.$$

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To prove that $\mathbf{AI} = \mathbf{A}$, note first that

$$\mathbf{I} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}, \tag{6}$$

where the jth column vector of $\mathbf{I}$ is the jth **basic unit vector**

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th entry}. \tag{7}$$

If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, then Fact 1 yields

$$\mathbf{Ae}_j = 0 \cdot \mathbf{a}_1 + \cdots + 1 \cdot \mathbf{a}_j + \cdots + 0 \cdot \mathbf{a}_n = \mathbf{a}_j. \tag{8}$$

Hence Fact 2 gives

$$\begin{aligned} \mathbf{AI} &= \mathbf{A}\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Ae}_1 & \mathbf{Ae}_2 & \cdots & \mathbf{Ae}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}; \end{aligned}$$

that is, $\mathbf{AI} = \mathbf{A}$. The proof that $\mathbf{IB} = \mathbf{B}$ is similar. (See Problems 41 and 42.)